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Circular coloring and Mycielski construction[☆]

Meysam Alishahi, Hossein Hajiabolhassan [*]

Department of Mathematical Sciences, Shahid Beheshti University, G.C., P.O. Box 1983963113, Tehran, Iran

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ABSTRACT

In this paper, we investigate the circular chromatic number of the iterated Mycielskian of graphs. It was shown by Simonyi and Tardos [G. Simonyi, G. Tardos, Local chromatic number, Ky Fan's theorem and circular colorings, Combinatorica 26 (5) (2006) 587–626] that the tth iterate of the Mycielskian of the Kneser graph $\mathrm{KG}(m, n)$ has the same circular chromatic number and chromatic number provided that $m + t$ is an even integer. We prove that if m is large enough, then $\chi(\mathrm{M}^t(\mathrm{KG}(m, n))) = \chi_c(\mathrm{M}^t(\mathrm{KG}(m, n)))$ where M^t is the tth iterate of the Mycielskian operator. Also, we consider the generalized Kneser graph $\mathrm{KG}(m, n, s)$ and show that there exists a threshold $m(n, s, t)$ such that $\chi(\mathrm{M}^t(\mathrm{KG}(m, n, s))) = \chi_c(\mathrm{M}^t(\mathrm{KG}(m, n, s)))$ for $m \geq m(n, s, t)$.

1. Introduction

Throughout this paper, we only consider finite graphs. A *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ implies that $f(u)f(v) \in E(H)$. In addition, if any edge in H is the image of some edge in G, then f is termed an onto-edge homomorphism. Also, if $f : V(G) \longrightarrow V(H)$ is a homomorphism and an onto map from $V(G)$ to $V(H)$, then f is called an onto-vertex homomorphism. Also, the symbol $\mathrm{Hom}(G, H)$ is used to denote the set of all homomorphisms from G to H. The *categorical product* $G \times H$ of graphs G and H is a graph whose vertex set is the Cartesian product $V(G) \times V(H)$; and any two vertices (u, u') and (v, v') are adjacent in $G \times H$ if u is adjacent with v and u' is adjacent with v'.

For a given graph G, the notation $g(G)$ stands for the girth of the graph G. We denote the neighborhood of a vertex $v \in V(G)$ by $N(v)$, and we let $N[v]$ denote the closed neighborhood of v, i.e., $N[v] = N(v) \cup \{v\}$. We denote by $[m]$ the set $\{1, 2, \ldots, m\}$, and by $\binom{[m]}{n}$ the collection of all n-subsets of $[m]$. The *Kneser graph* $\mathrm{KG}(m, n)$ is the graph with the vertex set $\binom{[m]}{n}$, in which A is adjacent to B if and only if $A \cap B = \varnothing$. It was conjectured by Kneser [14] in 1955, and proved by Lovász [16] in 1978, that $\chi(\mathrm{KG}(m, n)) = m - 2n + 2$ when $m \geq 2n$. A subset S of $[m]$ is called 2-*stable* if $2 \leq |x - y| \leq m - 2$ for all distinct elements x and y of S. The *Schrijver graph* $\mathrm{SG}(m, n)$ is the subgraph of $\mathrm{KG}(m, n)$ induced by all 2-stable n-subsets of $[m]$. It was proved by Schrijver [19] that $\chi(\mathrm{SG}(m, n)) = \chi(\mathrm{KG}(m, n))$, and every proper subgraph of $\mathrm{SG}(m, n)$ has chromatic number smaller than that of $\mathrm{SG}(m, n)$.

The so-called *generalized Kneser graphs* are generalized from the Kneser graphs in a natural way. Let m and n be positive integers with $m \geq 2n$. The generalized Kneser graph $\mathrm{KG}(m, n, s)$ is the graph whose vertex set is the set of n-subsets of $[m]$ with two n-subsets A and B are joined by an edge if $|A \cap B| \leq s$. Properties of generalized Kneser graphs have been studied in several papers; see [3,5,8].

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* Corresponding author. Tel.: +98 21 29902917; fax: +98 21 22431655.
E-mail addresses: m_alishahi@sbu.ac.ir (M. Alishahi), hhaji@sbu.ac.ir (H. Hajiabolhassan).

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The Erdős–Ko–Rado Theorem [6] states that if $\mathcal{F}$ is an independent set in the Kneser graph $\mathrm{KG}(m, n)$ with $m \geq 2n$, then $|\mathcal{F}| \leq \binom{m-1}{n-1}$. If $m > 2n$ and the equality holds, then $\mathcal{F}$ is *trivial*, by which we mean $|\bigcap_{F\in\mathcal{F}} F| = 1$. Here is a generalization of the Erdős–Ko–Rado Theorem.

Theorem A (*Wilson [22]*). *Let n and s be non-negative integers with $n > s$. If $m \geq (s+2)(n-s)$ and $\mathcal{F}$ is an independent set of the generalized Kneser graph $\mathrm{KG}(m, n, s)$, then $|\mathcal{F}| \leq \binom{m-s-1}{n-s-1}$.*

Ahlswede and Khachatrian [1] determined the maximum size of independent sets for all m.

Let G be a graph. A proper k-coloring of G is the partition of the vertex set into independent sets $V_1, \dots, V_k$. If n and d are positive integers with $n \geq 2d$, then the *circular complete graph* $K_{n,d}$ is the graph with the vertex set $\{0, 1, \dots, n-1\}$ in which i is connected to j if and only if $d \leq |i-j| \leq n-d$. A graph G is said to be (n, d)*-colorable* if G admits a homomorphism to $K_{n,d}$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the minimum of those ratios $\frac{n}{d}$ for which $\gcd(n, d) = 1$ and such that G admits a homomorphism to $K_{n,d}$. It can be shown that one need only consider onto-vertex homomorphisms [23].

It was shown that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ for every graph G [21,24], and hence $\chi(G) = \lceil\chi_c(G)\rceil$. Thus $\chi_c(G)$ is a refinement of $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$.

The circular chromatic number of Kneser graphs has been studied by Johnson, Holroyd, and Stahl [13]. They proved that $\chi_c(\mathrm{KG}(m, n)) = \chi(\mathrm{KG}(m, n))$ if $m \leq 2n+2$ or $n = 2$, and they conjectured that the equality holds for all Kneser graphs.

Conjecture 1 (*[13]*). *If $m \geq 2n+1$, then $\chi_c(\mathrm{KG}(m, n)) = \chi(\mathrm{KG}(m, n))$.*

This conjecture has been studied in several papers [4,10,17,20]. Hajiabolhassan and Zhu [10] proved that the conjecture holds if m is large enough. Later, it was shown that the conjecture holds when m is an even positive integer [17,20].

Theorem B (*[10]*). *If $m \geq 2n^2(n-1)$, then $\chi_c(\mathrm{KG}(m, n)) = \chi(\mathrm{KG}(m, n))$.*

The following notation will be used throughout the paper. Let G be a graph with the vertex set $\{v_1, \dots, v_n\}$. The Mycielskian $\mathrm{M}(G)$ of G is the graph defined on $\{v_1, \dots, v_n\} \cup \{v'_1, \dots, v'_n\} \cup \{z\}$ with the edge set $E(G) \cup \{v'_i v_j : v_i v_j \in E(G)\} \cup \{zv'_i : i \in [n]\}$. Mycielski [18] used this construction to increase the chromatic number of a graph while keeping the clique number fixed; $\chi(\mathrm{M}(G)) = \chi(G)+1$ and $\omega(\mathrm{M}(G)) = \omega(G)$. The vertex v'_i is the *twin* of the vertex v_i (v_i is also the *twin* of the vertex v'_i); and the vertex z is the *root* of $\mathrm{M}(G)$. If there is no ambiguity we shall always use z as the root of $\mathrm{M}(G)$. For $t \geq 2$, set $\mathrm{M}^t(G) = \mathrm{M}(\mathrm{M}^{t-1}(G))$. For $t \geq 2$, we define the set of roots of $\mathrm{M}^t(G)$ recursively. Let $\mathcal{R}(\mathrm{M}^{t-1}(G)) = \{z_1, \dots, z_{2^{t-1}-1}\}$ be the roots of $\mathrm{M}^{t-1}(G)$. Set

$$\mathcal{R}(\mathrm{M}^t(G)) = \mathcal{R}(\mathrm{M}^{t-1}(G)) \cup \{z'_1, \dots, z'_{2^{t-1}-1}\} \cup \{z\}$$

where z is the root and z'_i is the twin of z_i in $\mathrm{M}^t(G) = \mathrm{M}(\mathrm{M}^{t-1}(G))$.

The problem of calculating the circular chromatic number of the Mycielskian of graphs has been investigated in [2,7,9,12,15,20]. It turns out that the circular chromatic number of $\mathrm{M}(G)$ is not determined by the circular chromatic number of G alone. The problem of determining the circular chromatic number of the iterated Mycielskian of a complete graphs was discussed in [2]. It was conjectured in [2] that if $n \geq t+2 \geq 3$, then $\chi_c(\mathrm{M}^t(K_n)) = \chi(\mathrm{M}^t(K_n)) = n+t$.

Liu [15] prove that $\chi_c(\mathrm{M}^t(K_n)) = \chi(\mathrm{M}^t(K_n)) = n+t$ whenever $t \geq 2$ and $n \geq 2^{t-1}+2t-2$. This improves a similar result in [9]. Using the Borsuk–Ulam Theorem, Simonyi and Tardos [20] showed that if $n+t$ is even, then $\chi_c(\mathrm{M}^t(K_n)) = \chi(\mathrm{M}^t(K_n)) = t+n$.

In this paper, we study the behavior of the circular chromatic number under the Mycielski operation. In the next section, we introduce the concept of free coloring of graphs and explore its relationship to the circular chromatic number. Section 3 contains a generalization of the concept of free coloring. In Section 4, we introduce some sufficient conditions for the equality of the circular chromatic number and the chromatic number of graphs in terms of the (a, b)-free chromatic number. In the last section, we consider the generalized Kneser graph $\mathrm{KG}(m, n, s)$ and show that there exists a threshold $m(n, s, t)$ such that $\chi(\mathrm{M}^t(\mathrm{KG}(m, n, s))) = \chi_c(\mathrm{M}^t(\mathrm{KG}(m, n, s)))$ for $m \geq m(n, s, t)$. Also, we show that if $m \geq 2n^2(n-1) + \min\{2^{t+1}-2, 2^t+3\}n - \min\{0, 2n-t-3\}$, then $\chi(\mathrm{M}^t(\mathrm{KG}(m, n))) = \chi_c(\mathrm{M}^t(\mathrm{KG}(m, n)))$.

2. Free chromatic number

In this section, we introduce the concept of the free chromatic number of graphs. We show that if the free chromatic number of a graph G is large enough, then $\chi(G) = \chi_c(G)$.

Definition 1. An independent set in a graph G is a *free independent set* if it can be extended to at least two distinct maximal independent sets in G. □

Note that a vertex of a graph G is contained in a free independent set if and only if the graph obtained by deleting the closed neighborhood of this vertex has an edge. Furthermore, an independent set F in G is a free independent set of G if and only if there exists an edge $uv \in E(G)$ such that $(N(u) \cup N(v)) \cap F = \varnothing$.

Definition 2. If every vertex in a graph G belongs to some free independent set, then G is a *free graph*. $\square$

Definition 3. The *free chromatic number* of a graph G, denote by $\phi(G)$, is the minimum size of a partition of $V(G)$ into free independent sets. If G is not free, then we set $\phi(G) = \infty$. $\square$

The following simple lemma provides a necessary condition for the existence of graph homomorphism based on the free chromatic number of graphs.

Lemma 1. *Let G and H be connected free graphs. If there exists an onto-edge homomorphism from G to H, then $\phi(G) \leq \phi(H)$.*

An easy computation shows that if $\gcd(n, d) = 1$ and $d \geq 2$, then the free chromatic number of circular complete graph $K_{n,d}$ is less than twice its chromatic number.

Lemma 2. *Let G be a graph such that $\chi_c(G) = \frac{n}{d}$ with $\gcd(n, d) = 1$. If $d \geq 2$, or equivalently, if $\chi_c(G) \neq \chi(G)$, then $\phi(G) \leq \left\lceil \chi_c(G)\left(1 + \frac{1}{d-1}\right)\right\rceil \leq 2\chi(G) - 1$.*

Proof. Since the circular chromatic number of G is $\frac{n}{d}$, there exists a homomorphism from G to the circular complete graph $K_{n,d}$. It was proved [23] that for each i, the edge joining the vertices i and $i + d$ (in $K_{n,d}$) is the image of some two adjacent vertices of G under such a homomorphism. Consequently, the inverse image of any $d - 1$ consecutive vertices of $K_{n,d}$ is a free independent set of G. Hence, the free chromatic number of G is at most $\lceil \frac{n}{d-1} \rceil$. Note that $\lceil \frac{n}{d-1} \rceil = \lceil \frac{n}{d}\frac{d}{d-1} \rceil = \left\lceil \chi_c(G)\left(1 + \frac{1}{d-1}\right)\right\rceil \leq 2\chi(G) - 1$, which completes the proof. ■

The preceding lemma provides a sufficient condition for the equality of the chromatic number and the circular chromatic number of a graph. Hence, it can be of interest to have bounds for the free chromatic number of graphs.

Suppose G is a free graph with n vertices. Let $\bar{\alpha}(G)$ be the size of a largest free independent set in G. Note that $\phi(G) \geq \frac{n}{\bar{\alpha}(G)}$. Also, for any edge uv in G, let $d(uv) = |N(u) \cup N(v)|$. Define $d(G) = \min\{d(uv) : uv \in E(G)\}$. It is a simple matter to see that $d(G) \leq \Delta(G) + \delta(G)$. Note that $\bar{\alpha}(G) \leq n - d(G)$. This yields the next lemma.

Lemma 3. *Let G be a graph with n vertices. Then $\phi(G) \geq \frac{n}{\bar{\alpha}(G)} \geq \frac{n}{n-d(G)}$.*

The next theorem gives an upper bound for the free chromatic number of a graph G in terms of the chromatic number and $d(G)$.

Theorem 1. *If G is a free graph, then $\phi(G) \leq \chi(G) + d(G)$. In addition,*

(a) *If $g(G) \geq 5$, then $\phi(G) \leq \chi(G) + 4$.*
(b) *If $g(G) \geq 7$, then $\phi(G) \leq \chi(G) + 2$.*

Proof. Assume that uv is an edge in G such that $d(uv) = d(G)$. Properly color the vertices of the induced graph $G \setminus (N(u) \cup N(v))$ by using at most $\chi(G)$ colors. Assign to each vertex of $N(u) \cup N(v)$ a new color. The obtained coloring is a free coloring for G. Consequently, $\phi(G) \leq \chi(G) + d(G)$.

To prove Part (a), we first suppose that G is a tree. In view of the assumption, since G is a free graph, the diameter of G should be greater than 4. Let $u_1, \ldots, u_l$ be a longest path in G. Let V_1, V_2 form a proper 2-coloring for $G \setminus N[u_{l-1}]$. Set $V_3 = \{u_{l-1}\}$ and $V_4 = N(u_{l-1})$. One can check that V_1, V_2, V_3, V_4 is a free coloring for G. Now, suppose that G is not a tree. Let C be an induced cycle of G with length $g(G)$. Let $u_1, \ldots, u_g$ be the vertices of C in order. Set $V_1 = \{u_1\}$, $V_2 = \{u_2\}$, $V_3 = N(u_1) \setminus \{u_2\}$, and $V_4 = N(u_2) \setminus \{u_1\}$. Note that $N(u_1) \cap N(u_2) = \varnothing$ since $g(G) \geq 5$. Also, suppose that $V_5, V_6, \ldots, V_{\chi(G)+4}$ is a $\chi(G)$-coloring of $G \setminus (V_1 \cup V_2 \cup V_3 \cup V_4)$. It is easy to see that $V_1, V_2, \ldots, V_{\chi(G)+4}$ is a free coloring for G.

To prove Part (b), similarly, set $V_1 = N(u_1)$ and $V_2 = N(u_2)$ (note that by the proof of Part (a), we can assume that G is not a tree). Moreover, assume that $V_3, V_4, \ldots, V_{\chi(G)+2}$ is a proper $\chi(G)$-coloring of $G \setminus (V_1 \cup V_2)$. It is simple to see that $V_1, V_2, \ldots, V_{\chi(G)+2}$ is a free coloring for G. Thus, Part (b) follows. ■

In view of Theorem 1 and since $d(G) \leq \Delta(G) + \delta(G)$ we have the next corollary.

Corollary 1. *Let G be a free graph. Then $\phi(G) \leq \chi(G) + \Delta(G) + \delta(G)$.*

One naturally wonders by how much the free chromatic number can exceed the chromatic number. Let $K_n \times K_m$ be the categorical product of the complete graphs K_m and K_n. It is easily seen that the chromatic number of this graph is $\min\{m, n\}$. On the other hand, every independent set of size at least 2 can be extended to only one maximal independent set. Hence, if $m, n \geq 2$, then the free chromatic number of this graph is mn. In the special case $n = 2$ and $m \geq 2$, $\phi(K_2 \times K_m) = 2m = \chi(K_2 \times K_m) + d(K_2 \times K_m)$. Note that $K_2 \times K_m$ is $(m-1)$-regular bipartite graph and $d(K_2 \times K_m) = 2m - 2$. Therefore, the bound in the previous corollary is sharp.

By the proof of Theorem 1, if G is a graph that contains two adjacent vertices u and v such that the induced subgraph on $N(u) \cup N(v)$ is a free graph with girth greater than 4, then one can similarly show that $\phi(G) \leq \chi(G) + 4$. Hence, if G is a sparse graph, then Lemma 2 is not fruitful. This leads us to generalize the definition of free coloring.

3. Generalization of free coloring

In this section, we generalize the concept of free coloring in order to show that the circular chromatic number of the tth iterated Mycielskian of the generalized Kneser graph $\mathrm{KG}(m, n, s)$ is equal to its chromatic number whenever m is sufficiently large.

Definition 4. For a free independent set F, we say F is *supported* by the edge uv or uv *supports* F if $F \cap (N(u) \cup N(v)) = \varnothing$. The set of all edges that *support* F is denoted by $\mathrm{supp}(F)$. □

We next generalize the free chromatic number.

Definition 5. Let a and b be integers with $a \geq 0$ and $b \geq 1$. The (a, b)*-free chromatic number* of a graph G, denoted by $\phi_b^a(G)$, is the minimum natural number t (if there is no such t, then we define $\phi_b^a(G) = \infty$) such that the following conditions hold:

1. There exists a partition of the vertices of G into t independent sets such that all but at most a of them are free independent sets. For convenience, let the sets be $V_1, \ldots, V_t$ with $V_1, \ldots, V_{t-a}$ being free.
2. There exist edges $e_1, e_2, \ldots, e_{t-a}$ such that $e_i \in \mathrm{supp}(V_i)$ for $i \leq t - a$ and such that each vertex of G is incident with at most b edges in $e_1, e_2, \ldots, e_{t-a}$. □

If the partition $V_1, \ldots, V_{t-a}$ along with edges $e_1, e_2, \ldots, e_{t-a}$ satisfy the conditions 1 and 2 in Definition 5, then they form an (a, b)*-free coloring* of G. It should be noticed that there is no requirement that the edges $e_1, e_2, \ldots, e_{t-a}$ are distinct.

We next derive elementary properties of the (a, b)-free chromatic number of graphs. The first lemma is immediate.

Lemma 4. *Let G be a graph and $\alpha(G)$ be the independence number of G.*

(a) *For any positive integer b, $\phi_b^0(G) \geq \phi(G)$. Moreover, $\lim_{b\to\infty} \phi_b^0(G) = \phi(G)$.*
(b) *For integers a, a', b, b' such that $a' \geq a \geq 0$ and $b' \geq b \geq 1$, $\phi_b^a(G) \geq \phi_{b'}^{a'}(G)$.*
(c) *For integers a and b such that $a \geq 0$ and $b \geq 1$, $\phi_b^a(G) \geq \frac{|V(G)|-a\alpha(G)}{\alpha(G)}$.*

The proof of the next lemma is almost identical to that of Lemma 2; we omit it for brevity.

Lemma 5. *If there exist integers a and b with $a \geq 0$ and $b \geq 2$ such that $\phi_b^a(G) \geq 2\chi(G)$, then $\chi_c(G) = \chi(G)$.*

4. Mycielski construction and circular chromatic number

In this section, we obtain some relationships between the circular chromatic number and the (a, b)-free chromatic number of graphs. The following lemma was proved by Fan [7].

Lemma A ([7]). *Let $\mathrm{M}(G)$ be the Mycielski construction of G. Assume that z is the root. Also, for any $v \in V(G)$, let v' be the twin of v. If $\chi_c(\mathrm{M}(G)) = \frac{n}{d}$ with $\gcd(n, d) = 1$ and $d \geq 2$, then there is a homomorphism $c \in \mathrm{Hom}(\mathrm{M}(G), K_{n,d})$ such that $c(z) = 0$ and $c(v) = c(v')$ if $c(v) \notin [n - d + 1, d - 1] \pmod{n}$.*

Lemma 6. *Let G be a graph. If a and b are integers with $a \geq 0$ and $b \geq 1$, then $\phi_b^a(\mathrm{M}(G)) \geq \phi_{2b}^{a+b}(G)$. Consequently, for any positive integer t, $\phi_2^0(\mathrm{M}^t(G)) \geq \phi_{2^{t+1}}^{2^{t+1}-2}(G)$.*

Proof. Assume that $V(\mathrm{M}(G)) = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n', z\}$ where $V(G) = \{v_1, v_2, \ldots, v_n\}$, v_i' is the twin of v_i and z is the root of $\mathrm{M}(G)$. If $\phi_b^a(\mathrm{M}(G)) = \infty$, then there is nothing to prove. Suppose that $\phi_b^a(\mathrm{M}(G)) = t$. Let $V_1, V_2, \ldots, V_t$ along with $e_1, e_2, \ldots e_{t-a}$ be an (a, b)-free coloring for $\mathrm{M}(G)$ where $e_i \in \mathrm{supp}(V_i)$. Set $U_i = V_i \cap V(G)$.

Note that at most b edges of e_i's are incident with z. Without loss of generality, assume that e_i, for $i = 1, 2, \ldots, t - a - b$, is not incident with z. For i in $\{1, \ldots, t - a - b\}$, if $e_i = v_s v_k'$, then we set $e_i' = v_s v_k$, and otherwise we set $e_i' = e_i$. So e_i' supports U_i for any $1 \leq i \leq t - a - b$. Also, it is straightforward to check that for every vertex $v \in V(G)$ the number of edges among $e_1', e_2', \ldots, e_{t-a-b}'$ that are incident with v is less than or equal to $2b$. Therefore, $U_1, U_2, \ldots, U_t$ along with $e_1', e_2', \ldots, e_{t-a-b}'$ is an $(a + b, 2b)$-free coloring for G. The second assertion follows by induction on t. ■

Lemmas 5 and 6 yield the following corollary.

Corollary 2. *Let G be a graph. If $\chi_c(\mathrm{M}^t(G)) \neq \chi(\mathrm{M}^t(G))$, then $\phi_{2^{t+1}}^{2^{t+1}-2}(G) \leq 2\chi(\mathrm{M}^t(G)) - 1 = 2\chi(G) + 2t - 1$.*

The preceding corollary and the next lemma can be considered as generalizations of Lemma 2.

Lemma 7. *Let G be a graph and t be a positive integer. If $\chi_c(\mathrm{M}^t(G)) \neq \chi(\mathrm{M}^t(G))$, then $\phi_{2^{t+1}}^{2^t+3}(G) \leq 2\chi(\mathrm{M}^t(G)) - 1$.*

Proof. Suppose that $\chi_c(\mathrm{M}^t(G)) = \frac{n}{d}$, $\gcd(n, d) = 1$, and $d \geq 2$. In view of Corollary 2, the assertion holds if $t \leq 2$. Hence, assume that $t \geq 3$. Consider $\mathrm{M}^t(G)$ as the Mycielskian of $H = \mathrm{M}^{t-1}(G)$ and let $c \in \mathrm{Hom}(\mathrm{M}(H), K_{n,d})$ as in Lemma A. Assume that $n = k(d-1)+s$ where $1 \leq s \leq d-1$. Set $V_i = c^{-1}(\{(i-1)(d-1)+1, (i-1)(d-1)+2, \ldots, i(d-1)\})$ for $i = 1, 2, \ldots, k$ and $V_{k+1} = c^{-1}(\{k(d-1)+1, k(d-1)+2, \ldots, n\})$. It is known that for any i, the edge between vertices i and $i+d$ (in $K_{n,d}$) should be in the range of c. Hence, for any $i \in \{1, 2, \ldots, k+1\}$, the set $E_i = c^{-1}(\{(i-1)(d-1), i(d-1)+1\})$ is non-empty. For any $i \in \{1, 2, \ldots, k+1\}$, choose an $e_i \in E_i$. Note that for each $1 \leq j \leq n$, the number of edges among $e_1, e_2, \ldots, e_{k+1}$ which are incident with some vertices in $c^{-1}(j)$ is at most 2 (for $d > 2$ this number is at most 1). One can check that $V_1, V_2, \ldots, V_{k+1}$ along with $e_1, e_2, \ldots, e_{k+1}$ is a $(0, b)$-free coloring ($b \leq 2$) for $\mathrm{M}^t(G)$. Obviously, $2\chi(\mathrm{M}^t(G)) - 1 \geq k+1$. Now, we consider two cases:

Case (I) $d > 2$.

In this case, $V_1, V_2, \ldots, V_{k+1}$ along with $e_1, e_2, \ldots, e_{k+1}$ is a $(0, 1)$-free coloring and by Lemma 6 we have

$$k + 1 \geq \phi_1^0(\mathrm{M}^t(G)) \geq \phi_2^1(\mathrm{M}^{t-1}(G)) \geq \cdots \geq \phi_{2^t}^{2^t-1}(G).$$

On the other hand, we know that $\phi_{2^t}^{2^t-1}(G) \geq \phi_{2^{t+1}}^{2^t+3}(G)$.

Case (II) $d = 2$.

Assume that $\mathcal{R} = \{y_1, y_2, \ldots, y_{2^{t-1}-1}\} \cup \{y'_1, y'_2, \ldots, y'_{2^{t-1}-1}\} \cup \{z\}$ are the roots of $\mathrm{M}^t(G)$ where $T = \{y_1, y_2, \ldots, y_{2^{t-1}-1}\}$ are the roots of $H = \mathrm{M}^{t-1}(G)$ and $T' = \{y'_1, y'_2, \ldots, y'_{2^{t-1}-1}\}$ are the twins of the vertices of T (y'_i is the twin of y_i).

Set $U_i^{t-1} = V_i \cap V(\mathrm{M}^{t-1}(G))$, for $1 \leq i \leq k+1$. For any vertex $v \in V(\mathrm{M}^t(G))$, let $n(v)$ denote the number of edges among $e_1, e_2, \ldots, e_{k+1}$ which are incident with v. Without loss of generality, assume that e_i, for $i = 1, 2, \ldots, k+1-n(z)$, is not incident with z. If there is an $1 \leq i \leq k+1-n(z)$ such that $e_i = v_s v'_k$ where v_s and v_k are the vertices of H (v'_k is the twin of v_k), then we define $e_i^{t-1} = v_s v_k$, otherwise, set $e_i^{t-1} = e_i$. It is readily seen that e_i^{t-1} supports U_i^{t-1} for any $1 \leq i \leq k+1-n(z)$. Inductively, after iterating the aforementioned procedure $t-1$ times more, we obtain $U_1^0, U_2^0, \ldots, U_{k+1}^0$ along with $e_1^0, e_2^0, \ldots, e_{k+1-p}^0$ which is a $(p, 2^{t+1})$-free coloring for G. It is a simple matter to check that p is the number of edges among $e_1, e_2, \ldots, e_{k+1}$ which are incident with at least one vertex of $\mathcal{R}$.

Now, we show that $p \leq 2^t + 3$. To see this, let $S \subseteq T$ be the set of vertices whose colors (in the coloring c) are not in $[n-1, 1]$ (mod n). Define $S' = \{y'_j | y_j \in S\} \subseteq T'$. Note that c satisfies Lemma A, hence, for any $y_j \in S$, y_j and y'_j have the same color. Therefore, for every vertex $y_j \in S$, $n(y_j) + n(y'_j) \leq 2$. Also, it is easy to check that the number of edges among $e_1, e_2, \ldots, e_{k+1}$ which are incident with vertices in $(R \setminus S) \cup \{z\}$ is at most 5. Furthermore, for any vertex $v \in V(\mathrm{M}^t(G))$, $n(v) \leq 2$, consequently, $p \leq \sum_{y_i \in T} n(y_i) + \sum_{y'_i \in T'} n(y'_i) + n(z) \leq 2|S| + 2|T \setminus S| + 5 \leq 2^t + 3$. ■

4.1. Mycielski construction of generalized Kneser graphs

Here we investigate the circular chromatic number of the Mycielski construction of generalized Kneser graphs. Although, the exact value of $\chi(\mathrm{KG}(m, n, s))$ is unknown in general, we show that $\chi_c(\mathrm{M}^t(\mathrm{KG}(m, n, s))) = \chi(\mathrm{M}^t(\mathrm{KG}(m, n, s)))$ whenever m is large enough.

Theorem 2. *Let n, s and t be non-negative integers such that $n > s$. If m is large enough, then $\chi_c(\mathrm{M}^t(\mathrm{KG}(m, n, s))) = \chi(\mathrm{M}^t(\mathrm{KG}(m, n, s)))$.*

Proof. First, we show that $\chi(\mathrm{M}^t(\mathrm{KG}(m, n, s))) \leq \binom{m}{s+1} + t = O(m^{s+1})$. To see this, for every vertex $A \in V(\mathrm{KG}(m, n, s))$, choose an arbitrary subset $B \subseteq A$ of size $s+1$ and define $c(A) = B$. Obviously, c is a proper coloring for $\mathrm{KG}(m, n, s)$. Therefore, $\chi(\mathrm{KG}(m, n, s)) \leq \binom{m}{s+1}$, which implies that $\chi(\mathrm{M}^t(\mathrm{KG}(m, n, s))) \leq \binom{m}{s+1} + t$.

Now, we show that $\bar{\alpha}(\mathrm{KG}(m, n, s)) \leq \binom{2n}{s+2}\binom{m-s-2}{n-s-2} = O(m^{n-s-2})$. Let $\mathcal{F}$ be a free independent set in $\mathrm{KG}(m, n, s)$. Since $\mathcal{F}$ is a free independent set, there is an edge $AB \in E(\mathrm{KG}(m, n, s))$ such that $\mathcal{F} \cup \{A\}$ and $\mathcal{F} \cup \{B\}$ are still independent. Thus, for any $F \in \mathcal{F}$, $|F \cap A| \geq s+1$ and $|F \cap B| \geq s+1$, consequently, $|F \cap (A \cup B)| \geq s+2$. Now, by counting, the desired inequality holds.

If $\chi_c(\mathrm{M}^t(\mathrm{KG}(m, n, s))) \neq \chi(\mathrm{M}^t(\mathrm{KG}(m, n, s)))$, then by Corollary 2 we have

$$2\chi(\mathrm{M}^t(\mathrm{KG}(m, n, s))) - 1 \geq \phi_{2^{t+1}}^{2^{t+1}-2}(\mathrm{KG}(m, n, s)). \tag{1}$$

In view of Theorem A and Lemma 4(c), when $m \geq (s+2)(n-s)$ we have

$$\phi_{2^{t+1}}^{2^{t+1}-2}(\mathrm{KG}(m, n, s)) \geq \frac{\binom{m}{n} - (2^{t+1}-2)\binom{m-s-1}{n-s-1}}{\binom{2n}{s+2}\binom{m-s-2}{n-s-2}} = O(m^{s+2}).$$

Since $\chi(\mathrm{M}^t(\mathrm{KG}(m, n, s))) \leq \binom{m}{s+1} + t = O(m^{s+1})$, there exists a threshold $m(n, s, t)$ such that for $m \geq m(n, s, t)$, (1) does not hold, as desired. ■

The chromatic number of the generalized Kneser graph $\mathrm{KG}(m, n, 1)$ has been specified in [8]. The chromatic number of $\mathrm{KG}(m, n, s)$ remains open for $s \geq 2$. Motivated by the aforementioned theorem, we propose the following question.

Question 1. *Let m, n, and s be non-negative integers where $m > n > s$. Is it true that $\chi_c(\mathrm{KG}(m, n, s)) = \chi(\mathrm{KG}(m, n, s))$?*

It was proved by Hilton and Milner [11] that if X is an independent set of $\mathrm{KG}(m, n)$ of size

$$\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 2,$$

then

$$\bigcap_{A \in X} A = \{i\},$$

for some $i \in [m]$. Therefore, any independent set of size greater than $\binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1$ can be extended to a unique maximum independent set. This leads us to the following lemma.

Lemma 8. *Let m and n be positive integers with $m > 2n$. The size of any free independent set in the Kneser graph $\mathrm{KG}(m, n)$ is at most $\binom{m-1}{n-1} - \binom{m-n-1}{n-1}$. Also, for any non-negative integers a and b with $b \geq 1$, the (a, b)-free chromatic number of the Kneser graph $\mathrm{KG}(m, n)$ is at least*

$$\phi_b^a(\mathrm{KG}(m, n)) \geq \frac{\binom{m}{n} - a\binom{m-1}{n-1}}{\binom{m-1}{n-1} - \binom{m-n-1}{n-1}}.$$

Proof. Let $\mathcal{F}$ be a free independent set of $\mathrm{KG}(m, n)$. In view of Hilton and Milner Theorem, we should have $|\mathcal{F}| \leq \binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1$. On the contrary, assume that $|\mathcal{F}| = \binom{m-1}{n-1} - \binom{m-n-1}{n-1} + 1$. By considering Hilton and Milner Theorem and since $\mathcal{F}$ can be extended to two distinct maximal independent sets, there exist two positive integers i and j such that $\bigcap_{A \in X} A = \{i, j\}$. Hence, $|\mathcal{F}| \leq \binom{m-2}{n-2}$. On the other hand, it is easy to check that $\binom{m-2}{n-2} \leq \binom{m-1}{n-1} - \binom{m-n-1}{n-1}$ which is a contradiction. ■

The next theorem was shown in [10].

Theorem C ([10]). *For any positive integer n, if m is sufficiently large, then we have $\chi_c(\mathrm{SG}(m, n)) = \chi(\mathrm{SG}(m, n))$.*

Here is a generalization of the previous theorem.

Theorem 3. *For any non-negative integers n and t with $n \geq 1$, there is a threshold $m(n, t)$ such that $\chi_c(\mathrm{M}^t(\mathrm{SG}(m, n))) = \chi(\mathrm{M}^t(\mathrm{SG}(m, n))) = m - 2n + 2 + t$ whenever $m \geq m(n, t)$.*

Proof. Let $\chi_c(\mathrm{M}^t(\mathrm{SG}(m, n))) \neq m - 2n + 2 + t$. Set $k = \min\{2^{t+1} - 2,\ 2^t + 3\}$. By Corollary 2 and Lemma 7 we have

$$\phi_{2^{t+1}}^k(\mathrm{SG}(m, n)) \leq \phi_2^0(\mathrm{M}^t(\mathrm{SG}(m, n))) \leq 2(m - 2n + 2 + t) - 1. \tag{2}$$

On the other hand, the vertex set of $V(\mathrm{SG}(m, n))$ has cardinality $\binom{m-n-1}{n-1}\frac{m}{n} = O(m^n)$ (each 2-stable n-subsets of $[m]$ containing 1 corresponds to an integral solution of the equation $x_1 + x_2 + \cdots + x_n = m$ with $x_i \geq 2$. So there are $\binom{m-n-1}{n-1}$ 2-stable n-subsets of $[m]$ containing 1). In view of Lemma 8, we have

$$\phi_{2^{t+1}}^k(\mathrm{SG}(m, n)) \geq \frac{\binom{m-n-1}{n-1}\frac{m}{n} - k\binom{m-1}{n-1}}{\binom{m-1}{n-1} - \binom{m-n-1}{n-1}} = O(m^2).$$

Therefore, there is a threshold $m(n, t)$ such that if $m \geq m(n, t)$, then

$$\phi_2^0(\mathrm{M}^t(\mathrm{SG}(m, n))) \geq 2\chi(\mathrm{M}^t(\mathrm{SG}(m, n))) = 2(m - 2n + 2 + t).$$

This contradicts (2). ■

Here is a generalization of Theorem B.

Theorem 4. *For any non-negative integers n and t with $n \geq 1$, if $m \geq 2n^2(n-1) + \min\{2^{t+1}-2, 2^t+3\}n - \min\{0, 2n-t-3\}$, then $\chi_c(\mathrm{M}^t(\mathrm{KG}(m, n))) = \chi(\mathrm{M}^t(\mathrm{KG}(m, n)))$.*

Proof. The case $n = 1$ was proved in [10]. Hence, assume that $n \geq 2$. For convenience, set $k = \min\{2^{t+1} - 2, 2^t + 3\}$. In view of the proof of Theorem 3, it is sufficient to show that the following inequality holds for $m \geq 2n^2(n-1) + kn - \min\{0, 2n - t - 3\}$

$$\binom{m-1}{n-1} - \binom{m-n-1}{n-1} \leq \frac{\binom{m}{n} - k\binom{m-1}{n-1}}{2(m-2n+2+t)}.$$

By double counting we have

$$\binom{m-1}{n-1} - \binom{m-n-1}{n-1} \leq n\binom{m-2}{n-2}.$$

On the other hand, it is straightforward to check that

$$n\binom{m-2}{n-2} \leq \frac{\binom{m}{n} - k\binom{m-1}{n-1}}{2(m-2n+2+t)}$$

for $m \geq 2n^2(n-1) + kn - \min\{0, 2n - t - 3\}$. This completes the proof. ■

The next corollary is an improvement of Theorem B.

Corollary 3. *If $m \geq 2n^2(n-1) - 2n + 3$, then $\chi_c(\mathrm{KG}(m, n)) = \chi(\mathrm{KG}(m, n))$.*

Proof. The assertion holds for $n = 1$. Let $n \geq 2$. In view of the proof of Theorem 4, it is enough to show that

$$n\binom{m-2}{n-2} \leq \frac{\binom{m}{n}}{2(m-2n+2)}.$$

In other words, we should show that

$$2n^2(n-1)(m-2n+2) \leq m(m-1),$$

or equivalently,

$$2n^2(n-1)(3-2n) \leq (m-1)(m-2n^2(n-1)).$$

Set $f(m) = (m-1)(m-2n^2(n-1))$. It is straightforward to check that $f(2n^2(n-1)-2n+3) \geq 2n^2(n-1)(3-2n)$ and $f(m)$ is an increasing function for $m \geq 2n^2(n-1) - 2n + 3$. ■

It was shown in [2] that if G is a graph with chromatic number t, then we have $\chi_c(\mathrm{M}^{t-1}(G)) \leq \chi(\mathrm{M}^{t-1}(G)) - \frac{1}{2}$. Also, it was conjectured in [2] that if $n \geq t + 2 \geq 3$, then $\chi_c(\mathrm{M}^t(K_n)) = \chi(\mathrm{M}^t(K_n)) = n + t$.

Question 2. *Let m, n, and t be non-negative integers where $m > 2n$ and $0 \leq t \leq m - 2n$. Is it true that $\chi_c(\mathrm{M}^t(\mathrm{KG}(m, n))) = \chi(\mathrm{M}^t(\mathrm{KG}(m, n)))$?*

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